

Computational Hydraulics



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Numerical Solution of Ordinary Differential Equations

Module 6
(6 lectures)

Contents

- *Taylor series method*
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Introduction

- Numerical solution of ordinary differential equations is an important tool for solving a number of physical real world problems which are mathematically represented in terms of ordinary differential equations.
- Such as spring-mass system, bending of beams, open channel flows, pipe flows etc.
- The most of the scientific laws are represented in terms of ordinary differential equations, so to solve such systems we need efficient tools

Introduction

- If the differential equation contains derivatives of n th order, its called n th order differential equation.
- The solution of any differential equation should be such that it satisfies the differential equation along with certain initial conditions on the function.
- For the n th order equation, n independent initial conditions must be specified.

Introduction

- These equations can be solved analytically also, but those are limited to certain special forms of equations
- These equations can be linear or nonlinear.
- When the coefficients of these equations are constants, these are linear differential equations
- When the coefficients itself are functions of dependent variables, these are nonlinear differential equations

Introduction

- Numerical methods are not limited to such standard cases, it can be used to solve any physical situations.
- In numerical methods we get solution as a tabulation of values of the function at various values of the independent variable and data can be fit to some functional relationship, instead of exact functional relationship as in the analytical methods.
- The disadvantage of this method is that we have to re-compute the entire table if the initial conditions are changed

Introduction

- An equation of the form $dy/dx=f(x)$, with $f(x)$ given and with suitable initial conditions, say $y(a)$, also given can be integrated analytically or numerically by the methods discussed in the previous section, such as Simpson's 1/3 rule.

$$y(x) = y(a) + \int_a^x f(t)dt$$

- If $f(t)$ cannot be integrated analytically a numerical procedure can then be employed.
- The more general problem is nonlinear and of the form $dy/dx=f(x,y)$, f and $y(a)$ given, the problem is to find $y(x)$ for $x>a$

Taylor-series method

- Taylor series in which we expand y about the point $x=x_0$ is

$$y(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \frac{y'''(x_0)}{3!}(x - x_0)^3 + \dots$$

- If we assume $x - x_0 = h$
- Since $y(x_0)$ is initial condition, first term is known

$$y(x) = y(x_0) + y'(x_0)h + \frac{y''(x_0)}{2!}h^2 + \frac{y'''(x_0)}{3!}h^3 + \dots$$

- Error term of the Taylor series after the h^4 term can be written as

$$Error = \frac{y^{(v)}(\xi)}{5!}h^5,$$

where $0 < \xi < h$

Euler and modified Euler methods

- If derivative is complicated, Taylor series is not comfortable to use, error is difficult to determine
- Euler method uses first two terms of Taylor series, choosing h small enough to truncate the series after the first derivative term, then

$$y(x_0 + h) = y(x_0) + y'(x_0)h + \frac{y''(\xi)h^2}{2},$$

$$y_{n+1} = y_n + hy'_n + O(h^2).$$

Euler and modified Euler methods cont...

- Problem is lack of accuracy, requiring an extremely small step size
- If we use the arithmetic mean of the slopes at the beginning and end of the interval to compute y_{n+1} :

$$y_{n+1} = y_n + h \frac{y_n' + y_{n+1}'}{2}.$$

- This assumption gives us an improved estimate for y at x_{n+1} .
- y_{n+1}' can not be evaluated till the true value of y_{n+1} is known

Euler and modified Euler methods

- Modified Euler method predicts a value of y_{n+1} by simple Euler relation. It then uses this value to estimate y'_{n+1} giving an improved estimate of y_{n+1}
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- We need to re-correct y_{n+1} value till it makes the difference negligible
- We can find out the error in the modified Euler method by comparing with the Taylor series

Euler and modified Euler methods cont...

- This method is called Euler predictor-corrector method

$$y_{n+1} = y_n + y'_n h + \frac{1}{2} y''_n h^2 + \frac{y'''(\xi)}{6} h^3.$$

- Approximating y'' by forward difference, which has the error of $O(h)$:

$$y_{n+1} = y_n + h \left(y'_n + \frac{1}{2} \left[\frac{y'_{n+1} - y'_n}{h} + O(h) \right] h \right) + O(h^3),$$

$$y_{n+1} = y_n + h \left(y'_n + \frac{1}{2} y'_{n+1} - \frac{1}{2} y'_n \right) + O(h^3),$$

$$y_{n+1} = y_n + h \left(\frac{y'_n + y'_{n+1}}{2} \right) + O(h^3).$$

Runge-Kutta methods

- Fourth and fifth order Runge-Kutta methods
- Increment to the y is a weighted average of two estimates of the increment which can be taken as k_1 and k_2 .
- Thus for the equation $dy/dx=f(x,y)$

$$y_{n+1} = y_n + ak_1 + bk_2$$

$$k_1 = hf(x_n, y_n),$$

$$k_2 = hf(x_n + \alpha h, y_n + \beta k_1).$$

- We can think of the values k_1 and k_2 as estimates of the change in y when x advances by h , because they are the product of the change in x and a value for the slope of the curve, dy/dx .

Runge-Kutta methods cont...

- Uses Euler estimate of the first estimate of Δy , the other estimate is taken with x and y stepped up by the fractions α and β of h and of the earlier estimate of Δy , k_1
- Our problem is to devise a scheme of choosing the four parameters a, b, α, β . We do so by making

- Equations...

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2} f'(x_n, y_n) + \dots$$

- An equivalent form, since $df/dx = f_x + f_y dy/dx = f_x + f_y f$, is

$$y_{n+1} = y_n + hf_n + h^2 \left(\frac{1}{2} f_x + \frac{1}{2} f_y f \right)_n$$

Runge-Kutta methods cont...

- Fourth order Runge-Kutta methods are most widely used and are derived in similar fashion
- The local error term for the 4th order Runge-Kutta method is $O(h^5)$; the global error would be $O(h^4)$.
- Computationally more efficient than the modified Euler method, because while four evaluation of the function are required rather than two, the steps can be many fold larger for the same accuracy.

Runge-Kutta methods cont...

- The most commonly used set of values leads to the algorithm

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = hf(x_n, y_n),$$

$$k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right),$$

$$k_3 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right),$$

$$k_4 = hf(x_n + h, y_n + k_3),$$

Multi-step methods

- Runge-kutta type methods are called single step method
- When only initial conditions are available, ability to perform the next step with a different step size
- Uses past values of y and y' to construct a polynomial that approximates the derivative function, and extrapolate this into the next interval
- The number of past points that are used sets the degree of the polynomial and is therefore responsible for the truncation error.
- The order of the method is equal to the power of h in the global error term of the formula, which is also equal to one more than the degree of the polynomial.

Multi-step methods

- Adams method, we write the differential equation $dy/dx=f(x,y)$ in the form $dy=f(x,y)dx$, and we integrate between x_n and x_{n+1} :

$$\int_{x_n}^{x_{n+1}} dy = y_{n+1} - y_n = \int_{x_n}^{x_{n+1}} f(x, y) dx$$

- We approximate $f(x,y)$ as a polynomial in x , deriving this by making it fit at several past points
- Using 3 past points, approximate polynomial is quadratic, and for 4 points the polynomial is cubic
- More the past points, better the accuracy, until round-off error is negligible

Multi-step methods

- Suppose that we fit a second degree polynomial through the last three points (x_n, y_n) , (x_{n-1}, y_{n-1}) and (x_{n-2}, y_{n-2}) , we get a quadratic approximation to the derivative function:

$$f(x, y) = \frac{1}{2}h^2(f_n - 2f_{n-1} + f_{n-2})x^2 + \frac{1}{2}h(3f_n - 4f_{n-1} + f_{n-2})x + f_n$$

- Now we integrate between x_n and x_{n+1} . The result is a formula for the increment in y

$$y_{n+1} - y_n = \frac{h}{12}(23f_n - 16f_{n-1} + 5f_{n-2})$$

Multi-step methods

- We have the formula to advance y :

$$y_{n+1} = y_n + \frac{h}{12}[23f_n - 16f_{n-1} + 5f_{n-2}] + O(h^4)$$

- This formula resembles the single step formulas, in that the increment to y is a weighted sum of the derivatives times the step size, but differs in that past values are used rather than estimates in the forward direction.
- We can reduce the error by using more past points for fitting a polynomial

Multi-step methods

- In fact, when the derivation is done for four points to get a cubic approximation to $f(x,y)$, the following is obtained

$$y_{n+1} = y_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}] + O(h^5)$$

Multi-step methods

- Milne's method first predict a value for y_{n+1} by extrapolating the values for the derivative,
- Differs from Adam's method, as it integrates over more than one interval
- The required past values computed by Runge-Kutta or Taylor's series method.
- In this method, the four equi-spaced starting values of y are known, at the points x_n , x_{n-1} , x_{n-2} and x_{n-3}
- We may apply quadrature formula to integrate as follows

Multi-step methods

- Milne's method

$$\frac{dy}{dx} = f(x, y)$$

$$\int_{x_{n-3}}^{x_{n+1}} \left(\frac{dy}{dx} \right) dx = \int_{x_{n-3}}^{x_{n+1}} f(x, y) dx = \int_{x_{n-3}}^{x_{n+1}} P_2(x) dx$$

$$y_{n+1} - y_{n-3} = \frac{4h}{3} (2f_n - f_{n-1} + 2f_{n-2}) + \frac{28}{90} h^5 y^v(\xi_1)$$

- Where

$$x_{n-3} < \xi_1 < x_{n+1}$$

Multi-step methods

- The above predictor formula can be corrected by the following

$$\int_{x_{n-1}}^{x_{n+1}} \left(\frac{dy}{dx} \right) dx = \int_{x_{n-1}}^{x_{n+1}} f(x, y) dx = \int_{x_{n-1}}^{x_{n+1}} P_2(x) dx$$

$$y_{n+1,c} - y_{n-1} = \frac{h}{3} (f_{n+1} + 4f_n + f_{n-1}) - \frac{h^5}{90} y''(\xi_2)$$

- Where $x_{n-1} < \xi_2 < x_{n+1}$

Multi-step methods

- Adam-Moulton Method, more stable than and as efficient as Milne method .
- Adam-Moulton predictor formula:

$$y_{n+1} = y_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}] + \frac{251}{720}h^5 y''(\xi_1)$$

- Adam-Moulton corrector formula:

$$y_{n+1} = y_n + \frac{h}{24}[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}] - \frac{19}{720}h^5 y''(\xi_2)$$

- The efficiency of this method is about twice that of Runge-Kutta and Runge-kutta Fehlberg methods

Application to systems of equations and higher-order equations

- Generally any physical problems deals with a set of higher order differential equations. For example, the following equation represents a vibrating system in which a linear spring with spring constant k restores a displaced mass of weight w against a resisting force whose resistance is b times the velocity. The $f(x,t)$ is an external forcing function acting on the mass.

$$\frac{w}{g} \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = f(x, t)$$

System of equations and higher-order equations

- Reduce to a system of simultaneous first order equations
- For a second order equations the initial value of the function and its derivative are known i.e the n values of the variables or its derivatives are known, where n is the order of the system.
- When some of the conditions are specified at the boundaries of the specified interval, we call it a boundary value problem

Systems of equations and higher-order equations

- By solving for second derivative, we can normally express second order equation as

$$\frac{d^2x}{dt^2} = f\left(t, x, \frac{dx}{dt}\right),$$

$$x(t_0) = x_0,$$

$$x'(t_0) = x'_0$$

- The initial value of the function x and its derivatives are specified
- We convert to 1st order equation as

$$\frac{dx}{dt} = y,$$

$$x(t_0) = x_0,$$

Systems of equations and higher-order equations

- Then we can write

$$\frac{dy}{dt} = f(t, x, y),$$

$$y(t_0) = x_0'$$

- This pair of equations is equivalent to the original 2nd order equation
- For even higher orders, each of the lower derivatives is defined as a new function, giving a set of n first-order equations that correspond to an nth order differential equation.
- For a system of higher order equations, each is similarly converted, so that a larger set of first order equations results.

Systems of equations and higher-order equations

- Thus the n th order differential equation

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}),$$

$$y(x_0) = A_1,$$

$$y'(x_0) = A_2,$$

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$$y^{(n-1)}(x_0) = A_n$$

Systems of equations and higher-order equations

- Can be converted into a system of n first-order differential equations by letting $y_1 = y$ and

$$y_1' = y_2,$$

$$y_2' = y_3,$$

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$$y_{n-1}' = y_n,$$

$$y_n' = f(x, y_1, y_2, \dots, y_n);$$

Systems of equations and higher-order equations

- With initial conditions

$$y_1(x_0) = A_1,$$

$$y_2(x_0) = A_2,$$

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$$y_n(x_0) = A_n$$

- Now the Taylor-Series method, Euler Predictor-Corrector method, Runge-Kutta method, Runge-Kutta Fehlberg method, Adams-Moulton and Milne methods can be used to derive the various derivatives of the function

Examples of Open Channel Problems

- Steady flow through open channel

$$\rho V_s \frac{dV_s}{ds} + \frac{d}{ds}(p + \gamma z) = 0$$

Where p = pressure intensity

- Steady, uniform flow through open channel

$$\frac{d}{ds}(p + \gamma z) = 0$$

- The equation describing the variation of the flow depth for any variation in the bottom elevation is given by

$$\frac{dz}{dx} = (F_r^2 - 1) \frac{dy}{dx}$$

Examples of Open Channel Problems

- For gradually varied flow, variation of y with x

$$\frac{dy}{dx} = \frac{S_o - S_f}{1 - F_r^2}$$

- Or Gradually varied flow can be written as

$$\frac{dy}{dx} = \frac{S_o - S_f}{1 - (\alpha Q^2 B)/(gA^3)}$$

- For a very wide rectangular channel, $R \approx y$

$$\frac{dy}{dx} = \frac{gB(S_o C^2 B^2 y^3 - Q^2)}{C^2 (gBy^3 - \alpha BQ^2)}$$

Examples of Pipe Flow Problems

- Laminar flow, velocity distribution

$$u = \frac{r_0^2 - r^2}{4\mu} \left[-\frac{d}{ds}(p + \gamma z) \right]$$

- Time for flow establishment in a pipe

$$pA + \gamma AL \sin \alpha - \tau_0 \pi DL = \frac{d}{dt}(V \rho AL)$$

- Surge tank water-level Oscillations, the dynamic equation is

$$\frac{dQ}{dt} = \frac{gA_t}{L} (-z - cQ|Q|)$$

Assignments

1. Use the simple Euler method to solve for $y(0.1)$ from

$$\frac{dy}{dx} = x + y + xy \quad y(0) = 1$$

With $h=0.01$. Repeat this exercise with the modified Euler method with $h=0.025$. Compare the results.

2. Determine y at $x=0.2(0.2)0.6$ by the Runge-Kutta technique, given that

$$\frac{dy}{dx} = \frac{1}{x+y} \quad y(0) = 2$$

Assignments

3. Solve the following simultaneous differential equations by using

- (i) A fourth order Runge-Kutta method
- (ii) A fourth order Milne predictor-corrector algorithm

$$\frac{dy}{dx} = -x - yz, \frac{dz}{dx} = -y - xz, y(0) = 0, z(0) = 1.0$$

For $0.5 \geq x \geq 0.0$

4. Express the third order equation

$$y''' + ty'' - ty' - 2y = t, y(0) = y''(0) = 0, y'(0) = 1,$$

a set of first order equations and solve at $t = 0.2, 0.4, 0.6$ by the Runge-Kutta method ($h = 0.2$).

Assignments

5. Find y at $x=0.6$, given that

$$y'' = yy', y(0) = 1, y'(0) = -1$$

Begin the solution by the Taylor-series method, getting $y(0.1), y(0.2), y(0.3)$. Then advance to $x=0.6$ employing the Adams-Moulton technique with $h=0.1$ on the equivalent set of first-order equations.

6. Solve the pair of simultaneous equations by the modified Euler method for $t=0.2(0.2)0.6$. Recorrect until reproduced to three decimals.

$$\frac{dx}{dt} = xy + t, x(0) = 0, \frac{dy}{dt} = x - t, y(0) = 1,$$