Computational Hydraulics

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Solution of System of Linear and Non Linear Equations

Module 4

(4 lectures)

Contents

- ***Set of linear equations***
- ***Matrix notation***
- ***Method of solution:direct and iterative***
- ***Pathology of linear systems***
- ***Solution of nonlinear systems :Picard and Newton techniques***

Sets of linear equations

- Real world problems are presented through a set of simultaneous equations

$$F_1(x_1, x_2, \ldots, x_n) = 0$$

$$F_2(x_1, x_2, \ldots, x_n) = 0$$

$$.$$

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$$F_n(x_1, x_2, \ldots, x_n) = 0$$

- Solving a set of simultaneous linear equations needs several efficient techniques
- We need to represent the set of equations through matrix algebra

Matrix notation

Matrix : a rectangular array (n x m) of numbers

$$\mathbf{A} = [\mathbf{a_{ij}}] = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1m} \\ a_{21} & a_{22} & . & . & . & a_{2m} \\ . & & & & & . \\ . & & & & & . \\ . & & & & & . \\ a_{n1} & a_{n2} & . & . & . & a_{nm} \end{bmatrix}_{nxm}$$

Matrix Addition:

C = A+B = $[a_{ij}+ b_{ij}] = [c_{ij}]$, where

$$c_{ij} = a_{ij} + b_{ij}$$

Matrix Multiplication:

AB = C = $[a_{ij}][b_{ij}] = [c_{ij}]$, where

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

$i = 1,2,\ldots,n,$ $j = 1,2,\ldots,r.$

Matrix notation cont…

*AB ≠ BA

kA = C, where $c_{ij} = ka_{ij}$

A general relation for Ax = b is

$$b_i = \sum_{k=1}^{No.ofcols.} a_{ik}x_k, \quad i = 1,2,\ldots, No.ofrows$$

Matrix notation cont…

- Matrix multiplication gives set of linear equations as:

  $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1,$

  $a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2,$

  . . .

  . . .

  . . .

  $a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n,$

- In simple matrix notation we can write:

  Ax = b, where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1m} \\ a_{21} & a_{22} & . & . & . & a_{2m} \\ . & & & & & . \\ . & & & & & . \\ . & & & & & . \\ a_{n1} & a_{n2} & . & . & . & a_{nm} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix},$$

Matrix notation cont…

- Diagonal matrix (only diagonal elements of a square matrix are nonzero and all off-diagonal elements are zero)
- Identity matrix (diagonal matrix with all diagonal elements unity and all off-diagonal elements are zero)
- The order 4 identity matrix is shown below

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4.$$

Matrix notation cont…

- ***Lower triangular matrix***: if all the elements above the diagonal are zero

$$L = \begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{bmatrix}$$

- ***Upper triangular matrix***: if all the elements below the diagonal are zero

$$U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

- ***Tri-diagonal matrix***: if nonzero elements only on the diagonal and in the position adjacent to the diagonal

$$T = \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & d & e & 0 & 0 \\ 0 & f & g & h & 0 \\ 0 & 0 & i & j & k \\ 0 & 0 & 0 & l & m \end{bmatrix}$$

Matrix notation cont…

- Transpose of a matrix A (A^T): Rows are written as columns or vis a versa.

- Determinant of a square matrix A is given by:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det(A) = a_{11}a_{22} - a_{21}a_{12}$$

- Examples

$$A = \begin{bmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{4} \\ \mathbf{0} & \mathbf{2} & \mathbf{-3} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} \end{bmatrix}$$

$$A^T = \begin{bmatrix} \mathbf{3} & 0 & 1 \\ -1 & \mathbf{2} & 1 \\ 4 & -3 & \mathbf{2} \end{bmatrix}$$

Matrix notation cont…

- *Characteristic polynomial* $p_A(\lambda)$ and *eigenvalues* λ of a matrix:
- Note: eigenvalues are most important in applied mathematics
- For a square matrix A: we define $p_A(\lambda)$ as

$$p_A(\lambda) = |A - \lambda I| = det(A - \lambda I).$$

- If we set $p_A(\lambda) = 0$, solve for the roots, we get eigenvalues of A
- If A is n x n, then $p_A(\lambda)$ is polynomial of degree n
- *Eigenvector* w is a nonzero vector such that $Aw = \lambda w$, i.e., $(A - \lambda I)w = 0$

Methods of solution of set of equations

Direct methods are those that provide the solution in a finite and pre-determinable number of operations using an algorithm that is often relatively complicated. These methods are useful in linear system of equations.

Direct methods of solution

Gaussian elimination method

$$\begin{aligned} 4x_1 - 2x_2 + x_3 &= 15 \\ -3x_1 - x_2 + 4x_3 &= 8 \\ x_1 - x_2 + 3x_3 &= 13 \end{aligned}$$

Step1: Using Matrix notation we can represent the set of equations as

$$\begin{bmatrix} 4 & -2 & 1 \\ -3 & -1 & 4 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 8 \\ 13 \end{bmatrix}$$

Methods of solution cont…

- Step2: The Augmented coefficient matrix with the right-hand side vector

$$A \vdots b = \begin{bmatrix} 4 & -2 & 1 & \vdots & 15 \\ -3 & -1 & 4 & \vdots & 8 \\ 1 & -1 & 3 & \vdots & 13 \end{bmatrix}$$

- Step3: Transform the augmented matrix into Upper triangular form

$$\begin{bmatrix} 4 & -2 & 1 & 15 \\ -3 & -1 & 4 & 8 \\ 1 & -1 & 3 & 13 \end{bmatrix}, \qquad \begin{matrix} 3R_1 + 4R_2 \rightarrow \\ (-1)R_1 + 4R_3 \rightarrow \end{matrix} \begin{bmatrix} 4 & -2 & 1 & 15 \\ 0 & -10 & 19 & 77 \\ 0 & -2 & 11 & 37 \end{bmatrix}$$

$$2R_2 - 10R_3 \rightarrow \begin{bmatrix} 4 & -2 & 1 & 15 \\ 0 & -10 & 19 & 77 \\ 0 & 0 & -72 & -216 \end{bmatrix}$$

- Step4: The array in the upper triangular matrix represents the equations which after Back-substitution gives the solution the values of x_1, x_2, x_3

Method of solution cont…

- During the triangularization step, if a zero is encountered on the diagonal, we can not use that row to eliminate coefficients below that zero element, in that case we perform the *elementary row operations*
- we begin with the previous augmented matrix
- in a large set of equations multiplications will give very large and unwieldy numbers to overflow the computers register memory, we will therefore eliminate a_{i1}/a_{11} times the first equation from the i th equation

Method of solution cont…

- to guard against the zero in diagonal elements, rearrange the equations so as to put the coefficient of largest magnitude on the diagonal at each step. This is called ***Pivoting***. The diagonal elements resulted are called pivot elements. Partial pivoting , which places a coefficient of larger magnitude on the diagonal by row interchanges only, will guarantee a nonzero divisor if there is a solution of the set of equations.

- The round-off error (chopping as well as rounding) may cause large effects. In certain cases the coefficients sensitive to round off error, are called ***ill-conditioned matrix***.

Method of solution cont…

LU decomposition of A

- if the coefficient matrix A can be decomposed into lower and upper triangular matrix then we write: A=L*U, usually we get L*U=A′, where A′ is the permutation of the rows of A due to row interchange from pivoting
- Now we get det(L*U)= det(L)*det(U)=det(U)
- Then det(A)=det(U)

Gauss-Jordan method

- In this method, the elements above the diagonal are made zero at the same time zeros are created below the diagonal

Method of solution cont…

- Usually diagonal elements are made unity, at the same time reduction is performed, this transforms the coefficient matrix into an identity matrix and the column of the right hand side transforms to solution vector

- Pivoting is normally employed to preserve the arithmetic accuracy

Method of solution cont…

Example:Gauss-Jordan method

- Consider the augmented matrix as

$$\begin{bmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 6 & 1 & -6 & -5 & 6 \end{bmatrix}$$

- Step1: Interchanging rows one and four, dividing the first row by 6, and reducing the first column gives

$$\begin{bmatrix} 1 & 0.16667 & -1 & -0.83335 & 1 \\ 0 & 1.66670 & 5 & 3.66670 & -4 \\ 0 & -3.66670 & 4 & 4.33340 & -11 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

Method of solution cont…

- Step2: Interchanging rows 2 and 3, dividing the 2nd row by –3.6667, and reducing the second column gives

$$\begin{bmatrix} 1 & 0 & -1.5000 & -1.2000 & 1.4000 \\ 0 & 1 & 2.9999 & 2.2000 & -2.4000 \\ 0 & 0 & 15.0000 & 12.4000 & -19.8000 \\ 0 & 0 & -5.9998 & -3.4000 & 4.8000 \end{bmatrix}$$

- Step3: We divide the 3rd row by 15.000 and make the other elements in the third column into zeros

Method of solution cont…

$$\begin{bmatrix} 1 & 0 & 0 & 0.04000 & -0.58000 \\ 0 & 1 & 0 & -0.27993 & 1.55990 \\ 0 & 0 & 1 & 0.82667 & -1.32000 \\ 0 & 0 & 0 & 1.55990 & -3.11970 \end{bmatrix}$$

- Step4: now divide the 4th row by 1.5599 and create zeros above the diagonal in the fourth column

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -0.49999 \\ 0 & 1 & 0 & 0 & 1.00010 \\ 0 & 0 & 1 & 0 & 0.33326 \\ 0 & 0 & 0 & 1 & -1.99990 \end{bmatrix}$$

Method of solution cont…

Other direct methods of solution

Cholesky reduction (Doolittle's method)

- Transforms the coefficient matrix,A, into the product of two matrices, L and U, where U has ones on its main diagonal.Then *LU=A* can be written as

$$\begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{32} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} \\ 0 & 1 & u_{23} & u_{24} \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Method of solution cont…

- The general formula for getting the elements of L and U corresponding to the coefficient matrix for n simultaneous equation can be written as

$$l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj}$$ $j \le i,$ $i = 1,2,\ldots,n$ $l_{i1} = a_{i1}$

$$u_{ij} = \frac{a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj}}{l_{ii}}$$ $i \le j,$ $j = 2,3,\ldots,n.$ $u_{1j} = \frac{a_{1j}}{l_{11}} = \frac{a_{1j}}{a_{11}}$

Method of solution cont…

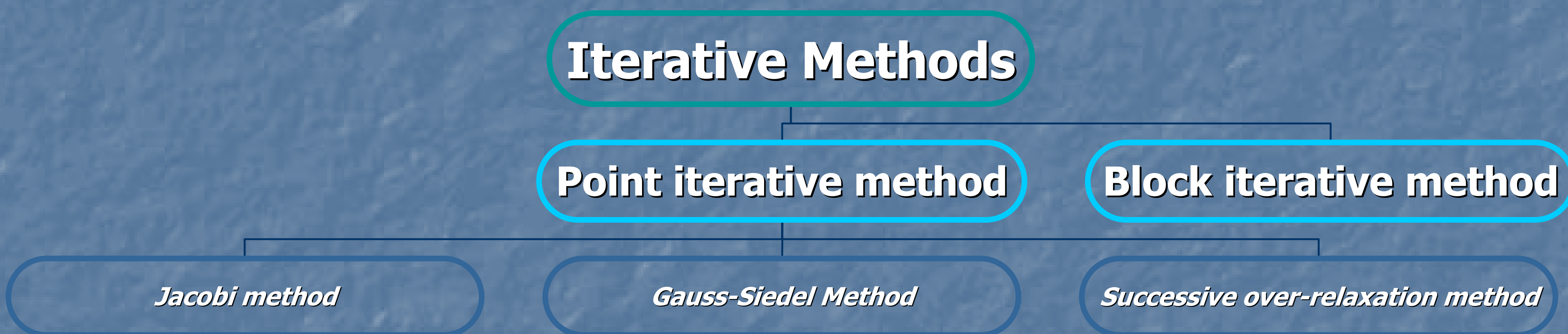
Iterative methods consists of repeated application of an algorithm that is usually relatively simple

Iterative method of solution

- coefficient matrix is sparse matrix (has many zeros), this method is rapid and preferred over direct methods,
- applicable to sets of nonlinear equations
- Reduces computer memory requirements
- Reduces round-off error in the solutions computed by direct methods

Method of solution cont…

- Two types of iterative methods: These methods are mainly useful in nonlinear system of equations.



Methods of solution cont…

Jacobi method

- Rearrange the set of equations to solve for the variable with the largest coefficient

Example:

$$6x_1 - 2x_2 + x_3 = 11,$$
$$x_1 + 2x_2 - 5x_3 = -1,$$
$$-2x_1 + 7x_2 + 2x_3 = 5.$$

$$x_1 = 1.8333 + 0.3333x_2 - 0.1667x_3$$
$$x_2 = 0.7143 + 0.2857x_1 - 0.2857x_3$$
$$x_3 = 0.2000 + 0.2000x_1 + 0.4000x_2$$

- Some initial guess to the values of the variables
- Get the new set of values of the variables

Methods of solution cont…

Jacobi method cont…

- The new set of values are substituted in the right hand sides of the set of equations to get the next approximation and the process is repeated till the convergence is reached

- Thus the set of equations can be written as

$$x_1^{(n+1)} = 1.8333 + 0.3333 x_2^{(n)} - 0.1667 x_3^{(n)}$$

$$x_2^{(n+1)} = 0.7143 + 0.2857 x_1^{(n)} - 0.2857 x_3^{(n)}$$

$$x_3^{(n+1)} = 0.2000 + 0.2000 x_1^{(n)} + 0.4000 x_2^{(n)}$$

Methods of solution cont…

Gauss-Siedel method

- Rearrange the equations such that each diagonal entry is larger in magnitude than the sum of the magnitudes of the other coefficients in that row (*diagonally dominant*)
- Make initial guess of all unknowns
- Then Solve each equation for unknown, the iteration will converge for any starting guess values
- Repeat the process till the convergence is reached

Methods of solution cont…

Gauss-Siedel method cont…

- For any equation *Ax=c* we can write

$$x_i = \frac{1}{a_{ii}}\left[c_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} x_j \right], \qquad i = 1,2,\ldots,n$$

- In this method the latest value of the x_i are used in the calculation of further x_i

Methods of solution cont…

Successive over-relaxation method

- This method rate of convergence can be improved by providing accelerators

- For any equation *Ax=c* we can write

$$\tilde{x}_i^{k+1} = \frac{1}{a_{ii}}\left[c_i - \sum_{j=1}^{i-1} a_{ij}x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij}x_j^{k}\right],$$

$$x_i^{k+1} = x_i^k + w(\tilde{x}_i^{k+1} - x_i^k) \qquad i = 1,2,\ldots,n$$

Methods of solution cont…

Successive over-relaxation method cont…

- Where $\tilde{x}_i^{k+1}$ determined using standard Gauss-Siedel algorithm

  k=iteration level,

  w=acceleration parameter (>1)

- Another form

$$x_i^{k+1} = (1-w)x_i^k + \frac{w}{a_{ii}}(c_i - \sum_{j=1}^{i-1} a_{ij}x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij}x_j^k)$$

Methods of solution cont…

Successive over-relaxation method cont..

Where 1<w<2: SOR method

0<w<1: weighted average Gauss Siedel method

- Previous value may be needed in nonlinear problems
- It is difficult to estimate w

Matrix Inversion

- Sometimes the problem of solving the linear algebraic system is loosely referred to as matrix inversion
- Matrix inversion means, given a square matrix [A] with nonzero determinant, finding a second matrix $[A^{-1}]$ having the property that $[A^{-1}][A]=[I]$, [I] is the identity matrix

$[A]x=c$

$x= [A^{-1}]c$

$[A^{-1}][A]=[I]=[A][A^{-1}]$

Pathology of linear systems

- Any physical problem modeled by a set of linear equations
- Round-off errors give imperfect prediction of physical quantities, but assures the existence of solution
- Arbitrary set of equations may not assure unique solution, such situation termed as "pathological"
- Number of related equations less than the number of unknowns, no unique solution, otherwise unique solution

Pathology of linear systems cont…

- Redundant equations (infinity of values of unknowns)

  $$x + y = 3, \quad 2x + 2y = 6$$

- Inconsistent equations (no solution)

  $$x + y = 3, \quad 2x + 2y = 7$$

- *Singular matrix* (n x n system, no unique solution)
- *Nonsingular matrix*, coefficient matrix can be triangularized without having zeros on the diagonal

Checking inconsistency, redundancy and singularity of set of equations:

- Rank of coefficient matrix (rank less than n gives inconsistent, redundant and singular system)

Solution of nonlinear systems

- Most of the real world systems are nonlinear and the representative system of algebraic equation are also nonlinear
- Theoretically many efficient solution methods are available for linear equations, consequently the efforts are put to first transform any nonlinear system into linear system
- There are various methods available for linearization

Method of iteration

- *Nonlinear system, example*: $x^2 + y^2 = 4; e^x + y = 1$
- Assume *x=f(x,y), y=g(x,y)*
- Initial guess for both x and y
- Unknowns on the left hand side are computed iteratively. Most recently computed values are used in evaluating right hand side

Solution of nonlinear systems

- Sufficient condition for convergence of this procedure is

$$\left|\frac{\partial f}{\partial x}\right| + \left|\frac{\partial f}{\partial y}\right| < 1 \qquad \left|\frac{\partial g}{\partial x}\right| + \left|\frac{\partial g}{\partial y}\right| < 1$$

- In an interval about the root that includes the initial guess
- This method depends on the arrangement of x and y i.e how $x=f(x,y)$, and $y=g(x,y)$ are written
- Depending on this arrangement, the method may converge or diverge

Solution of nonlinear systems

- The method of iteration can be generalized to n nonlinear equations with n unknowns. In this case, the equations are arranged as

$$x_1 = f_1(x_1, x_2, ..., x_n)$$

$$x_2 = f_2(x_1, x_2, ..., x_n)$$

$$.$$

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$$x_n = f_n(x_1, x_2, ..., x_n)$$

- A sufficient condition for the iterative process to converge is

$$\sum_{j=1}^{n} \left| \frac{\partial f_i}{\partial x_j} \right| < 1,$$

Newton technique of linearization

- Linear approximation of the function using a tangent to the curve
- Initial estimate x_0 not too far from the root
- Move along the tangent to its intersection with x-axis, and take that as the next approximation
- Continue till x-values are sufficiently close or function value is sufficiently near to zero
- Newton's algorithm is widely used because, at least in the near neighborhood of a root, it is more rapidly convergent than any of the other methods.
- Method is quadratically convergent, error of each step approaches a constant K times the square of the error of the previous step.

Newton technique of linearization

- The number of decimal places of accuracy doubles at each iteration
- Problem with this method is that of finding of *f′(x)*.
- First derivative *f′(x)* can be written as

$$\tan\theta = f'(x) = \frac{f(x_0)}{x_0 - x_1},$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

- We continue the calculation by computing

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

- In more general form,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$n = 0,1,2,\ldots$$

Newton-Raphson method

- F(x,y)=0, G(x,y)=0
- Expand the equation, using Taylor series about x_n and y_n

$$F(x_n + h, y_n + k) = 0 = F(x_n, y_n) + F_x(x_n, y_n)h + F_y(x_n, y_n)k$$

$$G(x_n + h, y_n + k) = 0 = G(x_n, y_n) + G_x(x_n, y_n)h + G_y(x_n, y_n)k$$

$$h = x_{n+1} - x_n, \; k = y_{n+1} - y_n$$

- Solving for h and k

$$h = \frac{GF_y - FG_Y}{F_x G_y - G_x F_y}; \qquad k = \frac{FG_x - GF_x}{F_x G_y - G_x F_y}$$

- Assume initial guess for x_n, y_n
- Compute functions, derivatives and x_n, y_n, h and k, Repeat procedure

Newton-Raphson method

- For n nonlinear equation

$$F_i(x_1+\Delta x_1, x_2+\Delta x_2+\ldots+x_n+\Delta x_n)=0$$
$$=F_i(x_1,x_2,\ldots,x_n)+\Delta x_1\frac{\partial F_i}{\partial x_1}+\Delta x_2\frac{\partial F_i}{\partial x_2}+\ldots+\Delta x_n\frac{\partial F_i}{\partial x_n},$$

$$i=1,2,3,\ldots,n$$

$$\frac{\partial F_1}{\partial x_1}\Delta x_1+\frac{\partial F_1}{\partial x_2}\Delta x_2+\ldots+\frac{\partial F_1}{\partial x_n}\Delta x_n=-F_1(x_1,x_2,\ldots,x_n)$$

$$\frac{\partial F_2}{\partial x_1}\Delta x_1+\frac{\partial F_2}{\partial x_2}\Delta x_2+\ldots+\frac{\partial F_2}{\partial x_n}\Delta x_n=-F_2(x_1,x_2,\ldots,x_n)$$

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$$\frac{\partial F_n}{\partial x_1}\Delta x_1+\frac{\partial F_n}{\partial x_2}\Delta x_2+\ldots+\frac{\partial F_n}{\partial x_n}\Delta x_n=-F_n(x_1,x_2,\ldots,x_n)$$

Picard’s technique of linearization

Nonlinear equation is linearized through:

- Picard’s technique of linearization
- Newton technique of linearization
- The *Picard's* method is one of the most commonly used scheme to solve the set of nonlinear differential equations.
- The *Picard's* method usually provide rapid convergence.
- A distinct advantage of the Picard's scheme is the simplicity and less computational effort per iteration than more sophisticated methods like Newton-Raphson method.

Picard's technique of linearization

- The general (parabolic type) equation for flow in a two dimensional, anisotropic non-homogeneous aquifer system is given by the following equation

$$\frac{\partial}{\partial x}\left[T_x \frac{\partial h}{\partial x}\right] + \frac{\partial}{\partial y}\left[T_y \frac{\partial h}{\partial y}\right] = S\frac{\partial h}{\partial t} + Q_p - R_r - R_s - Q_1$$

- Using the finite difference approximation at a typical interior node, the above ground water equation reduces to

$$B_{i,j}h_{i,j-1} + D_{i,j}h_{i-1,j} + E_{i,j}h_{i,j} + F_{i,j}h_{i+1,j} + H_{i,j}h_{i,j+1} = R_{i,j}$$

Picard's technique of linearization

- Where

$$B_{i,j} = -\frac{[T_{y_{i,j}} + T_{y_{i,j+1}}]}{2\Delta y^2}$$

$$D_{i,j} = -\frac{[T_{x_{i,j}} + T_{x_{i-1,j}}]}{2\Delta x^2}$$

$$F_{i,j} = -\frac{[T_{x_{i,j}} + T_{x_{i+1,j}}]}{2\Delta x^2}$$

$$H_{i,j} = -\frac{[T_{y_{i,j}} + T_{y_{i,j+1}}]}{2\Delta y^2}$$

Picard's technique of linearization

$$E_{i,j} = -(B_{i,j} + D_{i,j} + F_{i,j} + H_{i,j}) + \frac{S_{i,j}}{\Delta t}$$

$$R_{i,j} = \frac{S_{i,j} h_{0_{i,j}}}{\Delta t} - (Q)_{p_{i,j}} + (R)_{r_{i,j}} + (R)_{s_{i,j}}$$

- The Picard's linearized form of the above equation is given by

$$B^{n+1,m}{}_{i,j} h^{n+1,m+1}{}_{i,j-1} + D^{n+1,m}{}_{i,j} h_{i-1,j} + E^{n+1}{}_{,m_{i,j}} h_{i,j} +$$

$$F^{n+1,m}{}_{i,j} h_{i+1,j} + H^{n+1,m}{}_{i,j} h_{i,j+1} = R^{n+1,m}{}_{i,j}$$

Solution of Manning's equation by Newton's technique

- Channel flow is given by the following equation

$$Q = \frac{1}{n} S_o^{1/2} A R^{2/3}$$

- There is no general analytical solution to Manning's equation for determining the flow depth, given the flow rate as the flow area A and hydraulic radius R may be complicated functions of the flow depth itself..
- Newton's technique can be iteratively used to give the numerical solution
- Assume at iteration j the flow depth y_j is selected and the flow rate Q_j is computed from above equation, using the area and hydraulic radius corresponding to y_j

Manning's equation by Newton's technique

- This Q_j is compared with the actual flow Q
- The selection of y is done, so that the error

$$f(y_j) = Q_j - Q$$

- Is negligibly small
- The gradient of f w.r.t y is

$$\frac{df}{dy_j} = \frac{dQ_j}{dy_j}$$

- Q is a constant

Manning's equation by Newton's technique

- Assuming Manning's n constant

$$\left(\frac{df}{dy}\right)_j = \frac{1}{n} S_o^{1/2} \frac{d}{dy}\left(A_j R_j^{2/3}\right)$$

$$= \frac{1}{n} S_o^{1/2} \left(\frac{2AR^{-1/3}}{3} \frac{dR}{dy} + R^{2/3} \frac{dA}{dy} \right)_j$$

$$= \frac{1}{n} S_o^{1/2} A_j R_j^{2/3} \left(\frac{2}{3R} \frac{dR}{dy} + \frac{1}{A} \frac{dA}{dy} \right)_j$$

$$= Q_j \left(\frac{2}{3R} \frac{dR}{dy} + \frac{1}{A} \frac{dA}{dy} \right)_j$$

- The subscript j outside the parenthesis indicates that the contents are evaluated for $y=y_j$

Manning's equation by Newton's technique

- Now the Newton's method is as follows

$$\left(\frac{df}{dy}\right)_j = \frac{0 - f(y)_j}{y_{j+1} - y_j}$$

$$y_{j+1} = y_j - \frac{f(y_j)}{(df/dy)_j}$$

- Iterations are continued until there is no significant change in y, and this will happen when the error f(y) is very close to zero

Manning's equation by Newton's technique

- Newton's method equation for solving Manning's equation:

$$y_{j+1} = y_j - \frac{1 - Q/Q_j}{\left(\frac{2}{3R}\frac{dR}{dy} + \frac{1}{A}\frac{dA}{dy} \right)_j}$$

- For a rectangular channel $A=B_w y$, $R=B_w y/(B_w+2y)$ where B_w is the channel width, after the manipulation, the above equation can be written as

$$y_{j+1} = y_j - \frac{1 - Q/Q_j}{\left(\frac{5B_w + 6y_j}{3y_j(B_w + 2y_j)} \right)_j}$$

Assignments

1. Solve the following set of equations by Gauss elimination:

$$x_1 + x_2 + x_3 = 3$$
$$2x_1 + 3x_2 + x_3 = 6$$
$$x_1 - x_2 - x_3 = -3$$

Is row interchange necessary for the above equations?

2. Solve the system

$$9x + 4y + z = -17,$$
$$x - 2y - 6z = 14,$$
$$x + 6y = 4,$$

a. Using the Gauss-Jacobi method

b. Using the Gauss-Siedel method. How much faster is the convergence than in part (a).?

Assignments

3. Solve the following system by Newton's method to obtain the solution near (2.5,0.2,1.6)

$$x^2 + y^2 + z^2 = 9$$
$$xyz = 1$$
$$x + y - z^2 = 0$$

4. Beginning with (0,0,0), use relaxation to solve the system

$$6x_1 - 3x_2 + x_3 = 11$$
$$2x_1 + x_2 - 8x_3 = -15$$
$$x_1 - 7x_2 + x_3 = 10$$

Assignments

5. Find the roots of the equation to 4 significant digits using Newton-Raphson method

$$x^3 - 4x + 1 = 0$$

6. Solve the following simultaneous nonlinear equations using Newton-Raphson method. Use starting values $x_0 = 2,$ $y_0 = 0.$

$$x^2 + y^2 = 4$$
$$xy = 1$$