Computational Hydraulics

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Solution of Pipe Transients and Pipe Network Problems

Module 10

6 Lectures

Contents

- ***Basic equation of transients***
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Basic equations of transients

- The flow and pressures in a water distribution system do not remain constant but fluctuate throughout the day

- Two time scales on which these fluctuations occur

1. daily cycles
2. transient fluctuations

Basic equations of transients

- Continuity equation: applying the law of conservation of mass to the control volume (x_1 and x_2)

$$\int_{x_1}^{x_2} \frac{\partial}{\partial t}(\rho A)dx + (\rho A V)_2 - (\rho A V)_1 = 0$$

- By dividing throughout by Δx as it approach zero, the above equation can be written as

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho A V) = 0$$

- Expanding and rearranging various terms, using expressions for total derivatives, we obtain

$$\frac{1}{\rho}\frac{d\rho}{dt} + \frac{1}{A}\frac{dA}{dt} + \frac{\partial V}{\partial x} = 0$$

Basic equations of transients

- Now we define the bulk modulus of elasticity, K, of a fluid as $$K = \frac{dp}{\frac{d\rho}{\rho}}$$

- This can be written as $$\frac{d\rho}{dt} = \frac{\rho}{K}\frac{dp}{dt}$$

- Area of pipe, $A = \pi R^2$, where R is the radius of the pipe. Hence $dA/dt = 2\pi R dR/dt$. In terms of strain this may be written as $$\frac{dA}{dt} = 2A\frac{d\varepsilon}{dt}$$

- Now using hoop stress, we obtain $$\frac{d\varepsilon}{dt} = \frac{D}{2eE}\frac{dp}{dt}$$

Basic equations of transients

- Following the above equations one can write,

$$\frac{1}{A}\frac{dA}{dt} = \frac{D}{eE}\frac{dp}{dt}$$

- Substituting these equations into continuity equation and simplifying the equation yields $$\frac{\partial V}{\partial x} + \frac{1}{K}\left[1 + \frac{1}{eE/DK}\right]\frac{dp}{dt} = 0$$

- Let us define $a^2 = \dfrac{K/\rho}{1+(DK)/eE}$, where a is wave speed with which pressure waves travel back and forth.

- Substituting this expression we get the following continuity equation

$$\frac{\partial p}{\partial t} + V\frac{\partial p}{\partial x} + \rho a^2\frac{\partial V}{\partial x} = 0$$

Method of characteristics

- The dynamic and continuity equations for flow through a pipe line is given by

$$L1 = \frac{\partial Q}{\partial t} + gA\frac{\partial H}{\partial x} + \frac{f}{2DA}Q|Q| = 0$$

$$L2 = a^2\frac{\partial Q}{\partial x} + gA\frac{\partial H}{\partial t} = 0$$

Where Q=discharge through the pipe

H=piezometric head

A=area of the pipe

g=acceleration due to gravity

a=velocity of the wave

D=diameter of the pipe

x=distance along the pipe

t=time

Method of characteristics

- These equations can be written in terms of velocity

$$L1 = \frac{1}{g}\frac{\partial v}{\partial t} + \frac{\partial H}{\partial x} + \frac{f}{2Dg}v|v| = 0$$

$$L2 = \frac{\partial H}{\partial t} + \frac{a^2}{g}\frac{\partial v}{\partial x} = 0$$

- Where,

$$a = \sqrt{\frac{k}{e\left[1 + (kD/\rho E)\right]}}$$

Method of characteristics

- Where k=bulk modulus of elasticity

  ρ=density of fluid

  E=Young's modulus of elasticity of the material

Taking a linear combination of L1 and λL2, leads to

$$\left(\frac{\partial Q}{\partial t}+\lambda a^2\frac{\partial Q}{\partial x}\right)+\lambda gA\left(\frac{\partial H}{\partial T}+\frac{1}{\lambda}\frac{\partial H}{\partial x}\right)+\frac{f}{2DA}Q|Q|=0$$

- Assume *H=H(x,t);Q=Q(x,t)*

Method of characteristics

- Writing total derivatives ,

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x}\frac{dx}{dt} \qquad \frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}\frac{dx}{dt}$$

- Defining the unknown multiplier λ as

$$\frac{1}{\lambda} = \frac{dx}{dt} = \lambda a^2 \qquad \lambda = \pm\frac{1}{a}$$

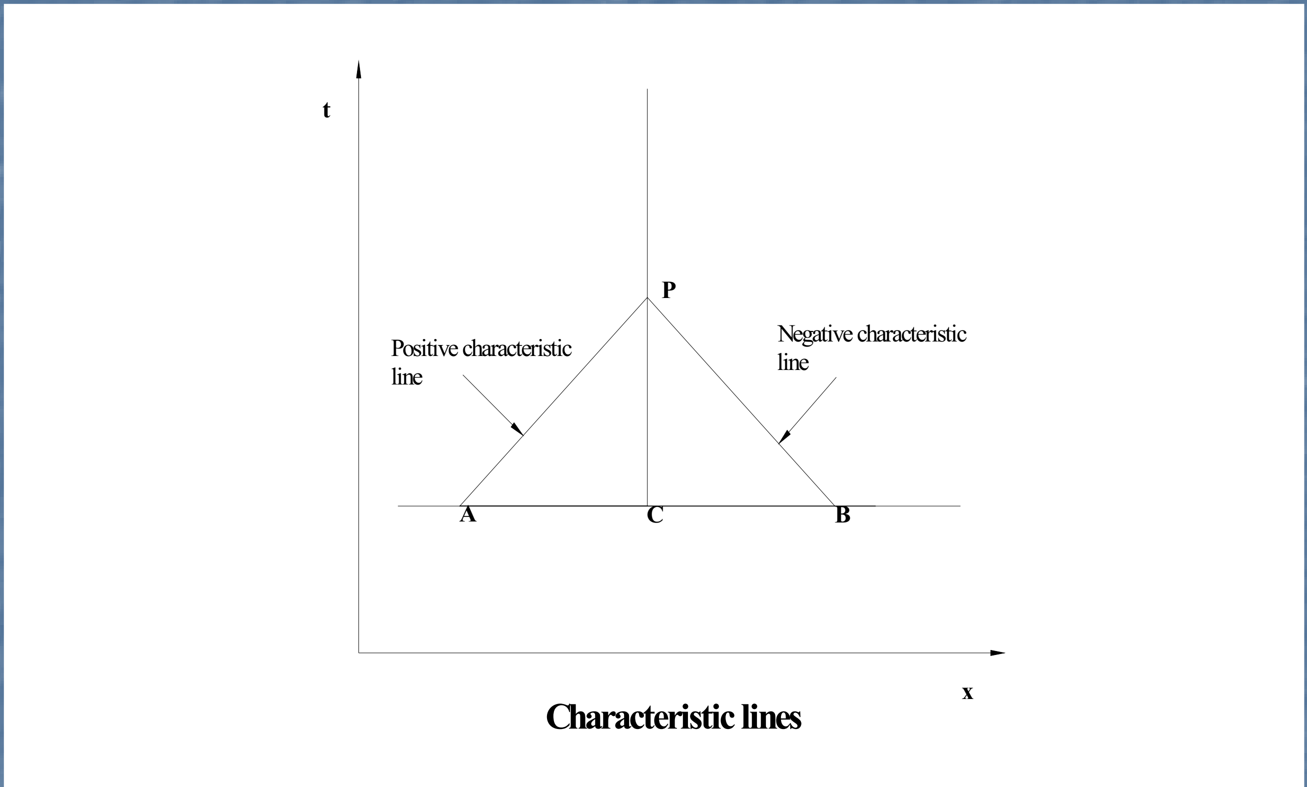
- Finally we get

$$\frac{dx}{dt} = \pm a \qquad \frac{dQ}{dt} \pm \frac{gA}{a}\frac{dH}{dt} + \frac{f}{2DA}Q\left|Q\right| = 0$$

- The above two equations are called characteristic equations and 2nd among them is condition along the characteristics

Method of characteristics

Figure…



- Constant head reservoir at *x=0*, at *x=L*, valve is instantaneously closed. Pressure wave travels in the upstream direction.

Complex boundary condition

- We may develop the boundary conditions by solving the positive or negative characteristic equations simultaneous with the condition imposed by the boundary.
- This condition may be in the form of specifying head, discharge or a relationship between the head and discharge
- Example: head is constant in the case of a constant level reservoir, flow is always zero at the dead end and the flow through an orifice is related to the head loss through the orifice.

Complex boundary condition

Constant-level upstream reservoir

- In this case it is assume that the water level in the reservoir or tank remains at the same level independent of the flow conditions in the pipeline
- This is true for the large reservoir volume
- If the pipe at the upstream end of the pipeline is 1, then $H_{P1,1} = H_{ru}$ where H_{ru} is the elevation of the water level in the reservoir above the datum.
- At the upstream end, we get the negative characteristic equation, $Q_{P1,1} = C_n + C_a H_{ru}$
-

Complex boundary condition

Constant-level downstream reservoir

- In this case, the head at the last node of pipe i will always be equal to the height of the water level in the tank above the datum, H_{rd}:

$$H_{Pi,n+1} = H_{rd}$$

- At the downstream end, we have the positive characteristic equation linking the boundary node to the rest of the pipeline. We can write

$$Q_{Pi,n+1} = Cp - C_a H_{rd}$$

Complex boundary condition

Dead end

- At a dead end located at the end of pipe i, the discharge is always zero:

$$Q_{Pi,n+1} = 0$$

- At the last node of pipe i, we have the positive characteristics equation. We get

$$H_{Pi,n+1} = \frac{C_p}{C_a}$$

Complex boundary condition

Downstream valve

- In the previous boundaries, either the head or discharge was specified,
- However for a valve we specify a relationship between the head losses through the valve and the discharge
- Denoting the steady-state values by subscript 0, the discharge through a valve is given by the following equation:

$$Q_0 = C_d A_{v0} \sqrt{2gH_0}$$

Complex boundary condition

Where

C_d=coefficient of discharge

A_{v0}=area of the valve opening

H_0=the drop in head

Q_0= a discharge

- By assuming that a similar relationship is valid for the transient state conditions, we get

$$Q_{Pi,n+1} = (C_d A_v)_P \sqrt{2gH_{Pi,n+1}}$$

- Where subscript P denotes values of Q and H at the end of a computational time interval

Complex boundary condition

- From the above two equations we can write

$$Q_{Pi,n+1}^2 = (Q_0 \tau)^2 \frac{H_{Pi,n+1}}{H_0}$$

- Where the effective valve opening is

$$\tau = (C_d A_v)_P / (C_d A_v)_0$$

- For the last section on pipe i, we have the positive characteristic equation

$$Q_{Pi,n+1}^2 + C_v Q_{Pi,n+1} - C_p C_v = 0$$

Complex boundary condition

- Where

$$C_v = (\tau Q_0)^2 / (C_a H_0)$$

- Solving for $Q_{Pi,n+1}$ and neglecting the negative sign with the radical term, we get

$$Q_{Pi,n+1} = 0.5(-C_v + \sqrt{C_v^2 + 4C_p C_v})$$

Pipe network problems

- The network designing is largely empirical.
- The main must be laid in every street along which there are properties requiring a supply.
- Mains most frequently used for this are 100 or 150mm diameter
- The nodes are points of junction of mains or where a main changes diameter.
- The demands along each main have to be estimated and are then apportioned to the nodes at each end in a ratio which approximated

Pipe network problems

There are a number of limitations and difficulties with respect to computer analysis of network flows , which are mentioned below:

1. The limitation with respect to the number of mains it is economic to analyze means that mains of 150 mm diameter and less are usually not included in the analysis of large systems, so their flow capacity is ignored
2. It is excessively time consuming to work out the nodal demands for a large system

Pipe network problems

1. The nodal demands are estimates and may not represent actual demands
2. Losses, which commonly range from 25% to 35% of the total supply, have to be apportioned to the nodal demands in some arbitrary fashion.
3. No diversification factor can be applied to the peak hourly demands representing reduced peaking on the larger mains since the total nodal demands must equal the input to the system
4. The friction coefficients have to be estimated.
5. No account is taken of the influence of pressure at a node on the demand at that node, I.e under high or low pressure the demand is assumed to be constant.

Governing Equation for Network Analysis

Every network has to satisfy the following equations:

1. Node continuity equations – the node continuity equations state that the algebraic sum of all the flows entering and leaving a node is zero.

$$\sum_{p\varepsilon\{j\}} Q(p) + \sum_{p\varepsilon\{j\}} Q(p) + C(j) = 0, \qquad j = 1,\ldots, NJ$$

Where NJ is the number of nodes, Q(p) is the flow in element p (m^3/s), C(j) is the consumption at node j (m^3/s), $p\varepsilon\{j\}$ refers to the set of elements connected to node j.

Network Analysis

2. Energy conservation equations – the energy conservation equations state that the energy loss along a path equals the difference in head at the starting node and end node of the path.

$$\sum_{p\varepsilon\{l\}}(\pm)h(p)+\sum_{p\varepsilon\{l\}}(\pm)h(p)-\left[H(s(l))-H(e(l))\right]=0 \quad l=1,\ldots,NL+NPATH$$

Where h(p) is the head loss in element p(m), s(l) is the starting node of path l, e(l) is the end of path 1, NL is the number of loops, and NPATH is the number of paths other than loops and $p\varepsilon\{l\}$ refers to the pipes belonging to path l. loop is a special case of path, wherein, the starting node and end node are the same, making the head loss around a loop zero, that is,

$$\sum(\pm)h(p)+\sum(\pm)h(p)=0$$

Network Analysis

3. Element characteristics – the equations defining the element characteristics relate the flow through the element to the head loss in the element. For a pipe element, h(p) is given by,

$$h(p) = R(p)Q(p)^e$$

Where R(p) is the resistance of pipe p and e is the exponent in the head loss equation. If Hazen-Williams equation is used, where e=1.852

$$R(p) = \frac{10.78L(p)}{D(p)^{4.87}CHW(p)^{1.852}}$$

Where L(p) is the length of pipe p(m), D(p) is the diameter of pipe p(m), and CHW (p) is the Hazen-Williams coefficient for pipe p.

Network Analysis

For a pump element, h(p) is negative as head is gained in the element. The characteristics of the pump element are defined by the head-discharge relation of the pump. This relationship may be expressed by a polynomial or in an alternate form. In this study, the following equation is used.

$$h(p) = -HR(m)\left[C1(m) - C2(m).\left[\frac{Q(p)}{QR(m)}\right]^{C3(m)}\right]$$

Where HR(m) is the rated head of the m-th pump (m), QR(m) is the rated discharge of m-th pump (m3/s), C1(m), C2(m) and C3(m) are empirical constants for the m-th pump obtained from the pump charateristics. Here p refers to the element corresponding to the m-th pump. If the actual pump characteristics are available, the constants C1, C2, C3 may be evaluated. C1 is determined from the shutoff head as

$$C1(m) = \frac{HO(m)}{HR(m)}$$

Network Analysis

Where HO(m) is the shutoff head of the m-th pump. As h(p)=-HR(m) for rated flow,

$$C1(m) - C2(m) = 1$$

From which C2(m)is determined. C3 (m) is obtained by fitting the equation to the actual pump characteristics.

For a pipe element,

$$Q(p) = \left[\frac{h(p)}{R(p)}\right]^{(1/e)} = \frac{H(i) - H(j)}{R(p)^{(1/e)} \left|H(i) - H(j)\right|^{(1-1/e)}}$$

For Hazen-Williams equation, the above equation becomes

$$Q(p) = \frac{H(i) - H(j)}{R(p)^{0.54} \left|H(i) - H(j)\right|^{0.46}}$$

Network Analysis

- Similarly for a pump element

$$Q(p) = (\pm)QR(m)\left[\frac{1}{C2(m)}\left[C1(m) \pm \frac{H(j)-H(i)}{HR(m)}\right]\right]^{\frac{1}{C3(m)}}$$

- Where outside the parenthesis, + sign is used if flow is towards node j and –sign is used if flow is away from node j and, inside the parenthesis, the + sign is used, if i is the node downstream of the pump and the – sign is used if j is the node downstream of the pump.

Network Analysis

- The network analysis problem reduces to one of solving a set of non-linear algebraic equations. Three types of formulation are used – the nodal, the path and the node and path formulation.
- Each formulation and method of analysis has its own advantages and limitations. In general path formulation with Newton-Raphson method gives the fastest convergence with minimum computer storage requirements.
- The node formulation is conceptually simple with a very convenient data base, but it has not been favoured earlier, because in conjunction with Newton-Raphson method, the convergence to the final solution was found to depend critically on the quality of the initial guess solution.
- The node and path formulation can have a self starting procedure without the need for a guess solution, but this formulation needs the maximum computer storage.

Node based models

The node (H) equations

- The number of equations to be solved can be reduced from L+J-1 to J by combining the energy equation for each pipe with continuity equation.
- The head loss equation for a single pipe can be written as

$$h = KQ^n$$

$$H_i - H_j = K_{ij}\left|Q_{ij}\right|^{n_{ij}} \operatorname{sgn} Q_{ij}$$

- Where H_i=head at i th node, L

  K_{ij}= head loss coefficient for pipe from node i to node j

  Q_{ij}= flow in pipe from node i to node j, L^3/t

  n_{ij}=exponent in head loss equation for pipe from i-j

Node based models

- The double subscript shows the nodes that are connect by a pipe
- Since the head loss is positive in the direction of flow, sgn Q_{ij}=sgn (H_i-H_j), and we solve for Q as

$$Q_{ij} = \mathrm{sgn}(H_i - H_j)(\left|H_i - H_j\right| / K_{ij})^{1/n_{ij}}$$

- The continuity equation at node I can be written as

$$\sum_{k=1}^{m_i} Q_{ki} = U_i$$

Where Q_{ki}=flow into node i from node k, L^3/T

U_i=consumptive use at node i, L^3/T

m_i=number of pipes connected to node i.

Node based models

- Combining energy and continuity equations for each flow in the continuity equation gives

$$\sum_{k=1}^{m_i} \operatorname{sgn}(H_k - H_i)\left(\frac{|H_k - H_i|}{K_{ki}}\right)^{1/n_{ki}} = U_i$$

- The above is a node H equation, there is one such equation for each node, and one unknown H_i for each equation

- These equations are all nonlinear

- The node (H) equations are very convenient for systems containing pressure controlled devices I.e. check valves, pressure reducing valves, since it is easy to fix the pressure at the downstream end of such a valve and reduce the value if the upstream pressure is not sufficient to maintain downstream pressure

Loop based models

The Loop (ΔQ) equations

- One approach is to setting up looped system problems is to write the energy equations in such a way that, for an initial solution, the continuity will be satisfied

- Then correct the flow in each loop in such a way that the continuity equations are not violated.

- This is done by adding a correction to the flow to every pipe in the loop .

- If there is negligibly small head loss, flow is added around the loop, if there is large loss, flow is reduced

- Thus the problem turns into finding the correction factor ΔQ such that each loop energy equation is satisfied

Loop based models

- The loop energy equations may be written

$$F(\Delta Q) = \sum_{i=1}^{m_l} K_i \left[\mathrm{sgn}(Qi_i + \Delta Q_l)\right] \left|Qi_i + \Delta Q_l\right|^n = dh_l \qquad (l=1,2,\dots,L)$$

Where

Qi_i = initial estimate of the flow in *i* th pipe, L^3/T

ΔQl = correction to flow in l th loop, L^3/T

m_l = number of pipes in l th loop

L = number of loops

Loop based models

- The Q_i terms are fixed for each pipe and do not change from one iteration to the next.
- The ΔQ terms refer to the loop in which the pipe falls
- The flow in a pipe is therefore $Q_i + \Delta Q$ for a pipe that lies in only one loop.
- For a pipe that lies in several loops (say ,a b, and c) the flow might be

$$Q_i + \Delta Q_a - \Delta Q_b + \Delta Q_c$$

Loop based models

- The negative sign in front of b term is included merely to illustrate that a given pipe may be situated in positive direction in one loop and in negative direction in another loop.

- When the loop approach is used, a total of L equations are required as there are l unknowns, one for each loop

Solution of pipe network problems through Newton-Raphson method

- Newton-Raphson method is applicable for the problems that can be expressed as *F(x)=0*, where the solution is the value of x that will force F to be zero
- The derivative of F can be a expressed by

$$\frac{dF}{dx} = \frac{F(x+\Delta x) - F(x)}{\Delta x}$$

- Given an initial estimate of x, the solution to the problem is the value of $x+\Delta x$ that forces F to 0. Setting $F(x+\Delta x)$ to zero and solving for Δx gives

$$\Delta x = -\frac{F(x)}{F^{'}(x)}$$

Solution of pipe network problems through Newton-Raphson method

- New value of $x+\Delta x$ becomes x for the next iteration. This process is continued until F is sufficiently close to zero
- For a pipe network problem, this method can be applied to the N-1=k, H-equations
- The head (H) equations for each node (1 through k), it is possible to write as:

$$F(H_i) = \sum_{j=1}^{m_i} \left[\operatorname{sgn}(H_j - H_i)\right]\left(\frac{|H_j - H_i|}{K_{ji}}\right)^{1/n_{ij}} - U_i = 0 \qquad (i = 1,2,\ldots,K)$$

- Where m_i= number of pipes connected to node I

  U_i= consumptive use at node i, L^3/T
- F(i) and F(i+1) is the value of F at ith and (i+1)th iteration, then

$$dF = F(i+1) - F(i)$$

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Solution of pipe network problems through Newton-Raphson method

- This change can also be approximated by total derivative

$$dF = \frac{\partial F}{\partial H_1}\Delta H_1 + \frac{\partial F}{\partial H2}\Delta H_2 + \ldots + \frac{\partial F}{\partial H_k}\Delta H_k$$

- Where *ΔH*= change in H between the ith and (i+1)th iterations, L
- Finding the values of *ΔH* which forces F(i+1)=0.
- Setting above two equations equal, results in a system of k linear equations with k unknowns *(ΔH)* which can be solved by the any linear methods

Solution of pipe network problems through Newton-Raphson method

- Initial guess for H
- Calculate partial derivatives of each F with respect to each H
- Solving the resulting system of linear equations to find H, and repeating until all of the F's are sufficiently close to 0
- The derivative of the terms in the previous equation is given by

$$\frac{d}{dH_j}\left[\operatorname{sgn}\left(H_i - H_j\right)\right]\left(\frac{\left|H_i - H_j\right|}{K_{ij}}\right)^{1/n_{ij}} = \frac{-1}{\left(n_{ij}\right)\left(K_{ij}\right)^{1/n_{ij}}}\left(H_i - H_j\right)^{(1/n_{ij})-1}$$

and

$$\frac{d}{dH_i}\left[\operatorname{sgn}\left(H_i - H_j\right)\right]\left(\frac{\left|H_i - H_j\right|}{K_{ij}}\right)^{1/n_{ij}} = \frac{1}{\left(n_{ij}\right)\left(K_{ij}\right)^{1/n_{ij}}}\left(H_i - H_j\right)^{(1/n_{ij})-1}$$

Solution of pipe network problems through Hardy-Cross method

- The linear theory method and the Newton-Raphson method can converge to the correct solution rapidly
- Manual solution or solution on small computers may not be possible with these methods
- However, the Hardy-cross method, which dates back to 1936, can be used for such calculations, in essence, the Hardy-Cross method is similar to applying the Newton-Raphson method to one equation at a time
- Hardy cross method is applied to ΔQ equations although it can be applied to the node equations and even the flow equations.
- The method, when applied to the ΔQ equations, requires an initial solution which satisfies the continuity equation

Solution of pipe network problems through Hardy-Cross method

- Nevertheless it is still widely used especially for manual solutions and small computers or hand calculators and produces adequate results for most problems
- For the l th loop in a pipe network the ΔQ equation can be written as follows

$$F(\Delta Q_1) = \sum_{i=1}^{m_l} K_i \left[\operatorname{sgn}\left(Qi_i + \Delta Q_l\right)\right] \left|Qi_i + \Delta Q_l\right|^n - dh_l = 0$$

Where

ΔQ_l=correction to l th loop to achieve convergence, L^3/T

Q_{ii}=initial estimates of flow in i th pipe (satisfies continuity),L^3/T

m_l=number of pipes in loop l

Solution of pipe network problems through Hardy-Cross method

- Applying the Newton-Raphson method for a single equation gives

$$\Delta Q(k+1) = \Delta Q - \frac{\sum_{i=1}^{m_l} K_i (Q_{i_i} + \Delta Q_l) \left| Q_{i_i} + \Delta Q_l \right|^{n-1}}{\sum_{i=1}^{m_l} K_i n_i \left| Q i_i + \Delta Q_l \right|^{n-1}}$$

- Where the k+1 refers to the values of ΔQ in the (k+1) th iteration, and all other values refer to the k th iterations and are omitted from the equation for ease of reading
- The above equation is equivalent to…

$$\Delta Q(k+1) = \Delta Q(k) - F(k) / F^{'}(k)$$

- Sign on the Q_i terms depend on how that pipe is situated in the loop under consideration.

Assignments

1. How many ΔQ equations must be set up for a network with L loops (and pseudo-loops), N nodes, and P pipes? How many H-equations must be set up?

2. What are the primary differences between the Hardy-Cross and Newton-Raphson method for solving the ΔQ equations?

3. For two pipes in parallel, with $K_1 > K_2$, what is the relationship between K_1, K_2, and K_e , the K for the equivalent pipe replacing 1 and 2 ($h=KQ^n$)?
   a. $K_1 > K_2 > K_e$
   b. $K_1 > K_e > K_2$
   c. $K_e > K_1 > K_2$

Assignments

4. Derive the following momentum equation by applying conservation of momentum for a control volume for transient flow through a pipe

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{fV\left|V\right|}{2D} = 0$$

5. Develop the system of equations for the following network (consists of 8 nodes and 9 elements, out of which 8 are pipe elements and the other is a pump element) to find the values of the specified unknowns. Also write a computer program to solve the system of equations.

Assignments continued