Computational Hydraulics

Indian Institute of Science
Bangalore, India

Prof. M.S.Mohan Kumar
Department of Civil Engineering

Introduction to Finite Difference Techniques

Module 7

6 lectures

Contents

- ***Types of finite difference techniques***
- ***Explicit and implicit techniques***
- ***Methods of solution***
- ***Application of FD techniques to steady and unsteady flows in open channels***

Types of FD techniques

- Most of the physical situation is represented by nonlinear partial differential equations for which a closed form solution is not available except in few simplified cases
- Several numerical methods are available for the integration of such systems. Among these methods, finite difference methods have been utilized very extensively
- Derivative of a function can be approximated by FD quotients.

Types of FD techniques

- Differential equation is converted into the difference equation

- Solution of difference equation is an approximate solution of the differential equation.

- Example: *f(x)* be a function of one independent variable x. assume at x_0, function be $f(x_0)$, then by using Taylor series expansion, the function $f(x_0+\Delta x)$ may be written as

$$f(x_0+\Delta x) = f(x_0) + \Delta x f'(x_0) + \frac{(\Delta x)^2}{2!} f''(x_0) + O(\Delta x)^3$$

Types of FD techniques

- $f'(x_0)=dy/dx$ at $x=x_0$
- $O(\Delta x)^3$: terms of third order or higher order of Δx
- Similarly $f(x_0- \Delta x)$ may be expressed as

$$f(x_0-\Delta x)=f(x_0)-\Delta x f'(x_0)+\frac{(\Delta x)^2}{2!}f''(x_0)+O(\Delta x)^3$$

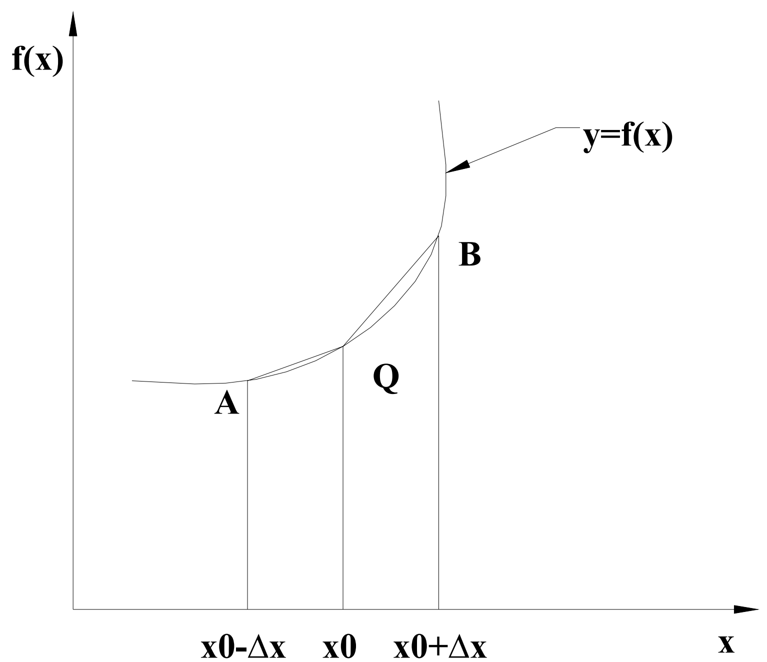
- Equation may be written as

$$f(x_0+\Delta x)=f(x_0)+\Delta x f'(x_0)+O(\Delta x)^2$$

- From this equation

$$\left.\frac{df}{dx}\right|_{x=x_{0_i}}=\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}+O(\Delta x)$$

Types of FD techniques



Finite Difference Approximation

Types of FD techniques

- Similarly

$$\left.\frac{df}{dx}\right|_{x=x_{0_i}} = \frac{f(x_0) - f(x_0 - \Delta x)}{\Delta x} + O(\Delta x)$$

- Neglecting O(Δx) terms in above equation we get
- Forward difference formula as given below

$$\left.\frac{df}{dx}\right|_{x=x_{0_i}} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

- Backward difference formula as shown below

$$\left.\frac{df}{dx}\right|_{x=x_{0_i}} = \frac{f(x_0) - f(x_0 - \Delta x)}{\Delta x}$$

- Both forward and backward difference approximation are first order accurate

Types of FD techniques cont…

- Subtracting the forward Taylor series From backward Taylor series, rearrange the terms, and divide by Δx

$$\left.\frac{df}{dx}\right|_{x=x_{0_i}} = \frac{f(x_0+\Delta x)-f(x_0-\Delta x)}{2\Delta x} + O(\Delta x)^2$$

- Neglecting the last term

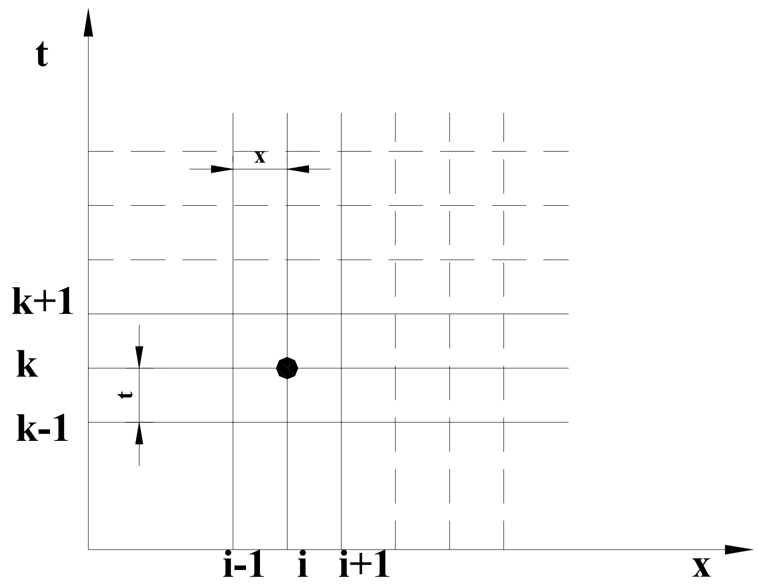
$$\left.\frac{df}{dx}\right|_{x=x_{0_i}} = \frac{f(x_0+\Delta x)-f(x_0-\Delta x)}{2\Delta x}$$

Types of FD techniques cont…

- This approximation is referred to as central finite difference approximation
- Error term is of order $O(\Delta x)^2$, known as second order accurate
- Central-difference approximations to derivates are more accurate than forward or backward approximations [$O(h^2)$ verses $O(h)$]
- Consider FD approximation for partial derivative

Types of FD techniques cont…

- Function *f(x,t)* has two independent variables, *x* and *t*
- Assume uniform grid size of Δx and Δt



Finite Difference Grid Approximation

Explicit and implicit techniques

- There are several possibilities for approximating the partial derivatives

- The spatial partial derivatives replaced in terms of the variables at the known time level are referred to as the *explicit* finite difference

- The spatial partial derivatives replaced in terms of the variables at the unknown time level are called *implicit* finite difference

- k is known time level and k+1 is the unknown time level. Then FD approximation for the spatial partial derivative , $\partial f/\partial x$, at the grid point (i,k) are as follows:

Explicit and implicit techniques

Explicit finite differences

Backward:

$$\frac{\partial f}{\partial x} = \frac{f_i^k - f_{i-1}^k}{\Delta x}$$

Forward:

$$\frac{\partial f}{\partial x} = \frac{f_{i+1}^k - f_i^k}{\Delta x}$$

Central:

$$\frac{\partial f}{\partial x} = \frac{f_{i+1}^k - f_{i-1}^k}{2\Delta x}$$

Explicit and implicit techniques

Implicit finite differences

Backward:

$$\frac{\partial f}{\partial x} = \frac{f_i^{k+1} - f_{i-1}^{k+1}}{\Delta x}$$

Forward:

$$\frac{\partial f}{\partial x} = \frac{f_{i+1}^{k+1} - f_i^{k+1}}{\Delta x}$$

Central:

$$\frac{\partial f}{\partial x} = \frac{f_{i+1}^{k+1} - f_{i-1}^{k+1}}{2\Delta x}$$

Explicit and implicit techniques

- By the known time level we mean that the values of different dependent variables are known at this time
- We want to compute their values at the unknown time level
- The known conditions may be the values specified as the initial conditions or they may have been computed during previous time step

Explicit finite difference schemes

- For the solution of hyperbolic partial differential equations, several explicit finite difference schemes have been proposed

- In the following section a number of typical schemes have been discussed which has its high relevance in hydraulic engineering

Unstable scheme

- For any unsteady situation, we can select the following finite-difference approximations:

Explicit finite difference schemes

- Approximations

$$\frac{\partial f}{\partial x} = \frac{f_{i+1}^{k} - f_{i-1}^{k}}{2\Delta x}$$

$$\frac{\partial f}{\partial t} = \frac{f_{i}^{k+1} - f_{i}^{k}}{\Delta t}$$

- In the above *f* refers to dependent variables
- Generally the finite difference scheme is inherently unstable; i.e., computation become unstable irrespective of the size of grid spacing, so the stability check is an important part of the numerical methods.

Explicit finite difference schemes

Diffusive scheme

- This scheme is slightly varying than the unstable scheme
- This method is easier to program and yields satisfactory results for typical hydraulic engineering applications. In this method the partial derivatives and other variables are approximated as follows:

$$\frac{\partial f}{\partial x} = \frac{f_{i+1}^{k} - f_{i-1}^{k}}{2\Delta x}$$

$$\frac{\partial f}{\partial t} = \frac{f_i^{k+1} - f^{*}}{\Delta t}$$

Explicit finite difference schemes

where

$$f^* = \frac{1}{2}(f_{i-1}^k - f_{i+1}^k)$$

- These approximations are applied to the conservation and non-conservation forms of the governing equations of the physical situations.

Explicit finite difference schemes

MacCormack Scheme

- This method is an explicit, two-step predictor-corrector scheme that is a second order accurate both in space and time and is capable of capturing the shocks without isolating them
- This method has been applied for analyzing one-dimensional, unsteady, open channel flows by various hydraulic engineers
- The general formulation for the scheme has been discussed as

Explicit finite difference schemes

MacCormack Scheme cont…

- Two alternative formulations for this scheme are possible. In the first alternative, backward FD are used to approximate the spatial partial derivatives in the predictor part and forward FD are utilized in the corrector part.

- The values of the variables determined during the predictor part are used during the corrector part

- In the second alternative forward FDs are used in the predictor and backward FD are used in the corrector part

Explicit finite difference schemes

MacCormack Scheme cont…

- Generally it is recommended to alternate the direction of differencing from one time step to the next

- The FD approximations for the first alternative of this scheme is given as follows. The ***predictor***

$$\frac{\partial f}{\partial t} = \frac{f_i^* - f_i^k}{\Delta t} \qquad \frac{\partial f}{\partial x} = \frac{f_i^k - f_{i-1}^k}{\Delta x}$$

Explicit finite difference schemes

MacCormack Scheme cont…

- The subscript * refers to variables computed during the predictor part
- The ***corrector***

$$\frac{\partial f}{\partial t} = \frac{f_i^{**} - f_i^k}{\Delta t} \qquad \frac{\partial f}{\partial x} = \frac{f_{i+1}^* - f_i^*}{\Delta x}$$

- the value of f_i at the unknown time level k+1 is given by

$$f_i^{k+1} = \frac{1}{2}(f_i^* + f_i^{**})$$

Explicit finite difference schemes

Lambda scheme

- In this scheme, the governing are is first transformed into λ-form and then discretize them according to the sign of the characteristic directions, thereby enforcing the correct signal direction.
- In an open channel flow, this allows analysis of flows having sub- and supercritical flows.
- This scheme was proposed by Moretti (1979) and has been used for the analysis of unsteady open channel flow by Fennema and Choudhry (1986)

Explicit finite difference schemes

Lambda scheme cont…

- Predictor

$$f_x^+ = \frac{2f_i^k - 3f_{i-1}^k + f_{i-2}^k}{\Delta x}$$

$$f_x^- = \frac{f_{i+1}^k - f_i^k}{\Delta x}$$

- Corrector

$$f_x^+ = \frac{f_i^* - f_{i-1}^*}{\Delta x}$$

$$f_x^- = \frac{-2f_i^* + 3f_{i-1}^* - f_{i-2}^*}{\Delta x}$$

Explicit finite difference schemes

- By using the above FD s and

$$\frac{\partial f}{\partial t} = \frac{f_i^{**} - f_i^k}{\Delta t}$$

- and using the values of different variables computed during the predictor part, we obtain the equations for unknown variables.

- The values at k+1 time step may be determined from the following equations:

$$f_i^{k+1} = \frac{1}{2}(f_i^{*} + f_i^{**})$$

Explicit finite difference schemes

Gabutti scheme

- This is an extension of the Lambda scheme. This allows analysis of sub and super critical flows and has been used for such analysis by Fennema and Chaudhry (1987)
- The general formulation for this scheme is comprised of predictor and corrector parts and the predictor part is subdivided into two parts
- The λ-form of the equations are used the partial derivatives are replaced as follows:

Explicit finite difference schemes

Gabutti scheme cont…

- Taking into consideration the correct signal direction

Predictor:

Step1: spatial derivatives are approximated as follows:

$$f_x^+ = \frac{f_i^k - f_{i-1}^k}{\Delta x} \qquad f_x^- = \frac{f_{i+1}^k - f_i^k}{\Delta x}$$

Explicit finite difference schemes

Gabutti scheme cont…

- By substituting

$$\frac{\partial f}{\partial t} = \frac{f_i^{**} - f_i^k}{\Delta t}$$

- Step2: in this part of the predictor part we use the following finite-difference approximations:

$$f_x^+ = \frac{2 f_i^k - 3 f_{i-1}^k + f_{i-2}^k}{\Delta x}$$

$$f_x^- = \frac{-2 f_i^k + 3 f_{i-1}^k - f_{i-2}^k}{\Delta x}$$

Explicit finite difference schemes

Gabutti scheme cont…

Corrector: in this part the predicted values are used and the corresponding values of coefficients and approximate the spatial derivatives by the following finite differences:

$$f_x^+ = \frac{f_i^* - f_{i-1}^*}{\Delta x} \qquad f_x^- = \frac{f_{i+1}^* - f_i^*}{\Delta x}$$

- The values at k+1 time step may be determined from the following equations:

$$f_i^{k+1} = f_i^k + \frac{1}{2}\Delta t(f_i^* + f_i^{**})$$

Implicit finite difference schemes

- In this scheme of implicit finite difference, the spatial partial derivatives and/or the coefficients are replaced in terms of the values at the unknown time level

- The unknown variables are implicitly expressed in the algebraic equations, this methods are called implicit methods.

- Several implicit schemes have been used for the analysis of unsteady open channel flows. The schemes are discussed one by one.

Implicit finite difference schemes

Preissmann Scheme

- This method has been widely used
- The advantage of this method is that the variable spatial grid may be used
- Steep wave fronts may be properly simulated by varying the weighting coefficient
- This scheme also yields an exact solution of the linearized form of the governing equations for a particular value of Δx and Δt.

Implicit finite difference schemes

Preissmann Scheme cont…

- General formulation of the partial derivatives and other coefficients are approximated as follows:

$$\frac{\partial f}{\partial t} = \frac{(f_i^{k+1} + f_{i+1}^{k+1}) - (f_i^k + f_{i+1}^k)}{2\Delta t}$$

$$\frac{\partial f}{\partial x} = \frac{\alpha(f_{i+1}^{k+1} - f_i^{k+1})}{\Delta x} + \frac{(1-\alpha)(f_{i+1}^k - f_i^k)}{\Delta x}$$

$$f = \frac{1}{2}\alpha(f_{i+1}^{k+1} + f_i^{k+1}) + \frac{1}{2}(1-\alpha)(f_{i+1}^k + f_i^k)$$

Implicit finite difference schemes

Preissmann Scheme

- Where α is a weighting coefficient and f refers to unknown variables and coefficients.
- By selecting a suitable value for α, the scheme may be made totally explicit (α=0) or implicit (α=0)
- The scheme is stable if 0.55< $\alpha \leq 1$

Assignments

1. A large flat steel plate is 2 cm thick. If the initial temperature within the plate are given, as a function of the distance from one face, by the equations

$u = 100x$ for $0 \le x \le 1$

$u = 100(2 - x)$ for $0 \le x \le 1$

Find the temperatures as a function of x and t if both faces are maintained at 0 degree centigrade. The one dimensional heat flow equation is given as follows

$$\frac{\partial u}{\partial t} = \frac{k}{c\rho}\frac{\partial^2 u}{\partial x^2}$$

Take *k=0.37 cρ=0.433.*

Assignments

2. Solve for the temperature at *t=2.06* sec in the 2-cm thick steel slab of problem (1) if the initial temperatures are given by

$$u(x,0) = 100 \sin\left(\frac{\pi x}{2}\right)$$

Use the explicit method with *Δx=0.25* cm. compare to the analytical solution:

$$100 e^{-0.3738 t} \sin(\pi x / 2)$$

3. Using Crank-Nicolson method, solve the following equation

$$k \frac{\partial^2 u}{\partial x^2} - c\rho \frac{\partial u}{\partial t} = f(x)$$

Solve this when $f(x) = x(x-1)$ subject to conditions

$$u(0,t) = 0, u(1,t) = 0, u(x,0) = 0.$$

Take *Δx=0.2, k=0.37 cρ=0.433.* solve for five time steps.